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B. Sc. (Honrs) Part 1paper 1

Subject:Mathematics

Title/Heading of topic:Fundamental theorem of Algebra

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Fundamental theorem of Algebra

Every polynomial equation with real coefficients has at least one root

Every equation $f(x) = 0$ of the n^{th} degree has n roots

Let $f(x)$ be the polynomial $a_0x^n + a_1x^{n-1} + \ldots + a_n$.

We assume that every equation $f(x) = 0$ has at least one root real or imaginary

Let α_1 be a root of $f(x) = 0$.

Then $f(x)$ is exactly divisible by $x - \alpha_1$, so that

$$f(x) = (x - \alpha_1)\,\phi_1(x)$$

where $\phi_1(x)$ is a rational integral function of degree $n - 1$.

Again $\phi_1(x) = 0$ has a root real or imaginary and let that root be a_2.

Then $\phi_1(x)$ is exactly divisible by $x - \alpha_2$, so that

$$\phi_1(x) = (x - \alpha_2)\,\phi_2(x)$$

where $\phi_2(x)$ is a rational integral function of degree $n - 2$.

$$\therefore f(x) = (x - \alpha_1)(x - \alpha_2)\,\phi_2(x).$$

By continuing in this way, we obtain

$$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)\,\phi_n(x)$$

where $\phi_n(x)$ is of degree $n - n$, i.e., zero

$\therefore \phi_n(x)$ is a constant.

Equating the coefficients of x^n on both sides we get

$$\phi_n(x) = \text{coefficients of } x^n$$

$$= a_0$$

$\therefore f(x) = a_0(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$.

Hence the equation $f(x) = 0$ has n roots, since $f(x)$ vanished when x has any one of the values $\alpha_1, \alpha_2, \ldots \alpha_n$. If x is given any value different from any one of these n roots, then no factor of $f(x)$ can vanish and the equation is not satisfied. Hence $f(x) = 0$ cannot have more than n roots.

In an equation with rational coefficients, imaginary roots occur in pairs.

Let the equation be $f(x) = 0$ and let $\alpha + i\beta$ be an imaginary root of the equation. We shall show that $\alpha - i\beta$ is also a root.

We have $(x - \alpha - i\beta)(x - \alpha + i\beta) = (x - \alpha)^2 + \beta^2$ ....(1)

If $f(x)$ is divided by $(x - \alpha)^2 + \beta^2$, let the quotient be $Q(x)$ and the remainder be $Rx + R'$

Here $Q(x)$ is of degree $(n - 2)$.

$\therefore f(x) = \{(x - \alpha)^2 + \beta^2\} Q(x) + Rx + R'$ ....(2)

Substituting $(\alpha + i\beta)$ for x in the equation (2), we get

$$f(\alpha + i\beta) = \{(\alpha + i\beta - \alpha)^2 + \beta^2\} Q(\alpha + i\beta) + R(\alpha + i\beta) + R'$$

$$= R(\alpha + i\beta) + R'$$

But $f(\alpha + i\beta) = 0$ since $\alpha + i\beta$ is a root of $f(x) = 0$.

Therefore

$$R(\alpha + i\beta) + R' = 0.$$

Equating to zero the real and imaginary parts

$$R\alpha + R' = 0 \text{ and } R\beta = 0.$$

Since $\beta \neq 0$, R = 0 and so R' = 0

$\therefore f(x) = \{(x - \alpha)^2 + \beta^2\}Q(x)$.

$\therefore \alpha - i\beta$ is also a root of f(x) = 0.

Solved Problems

1. Form a rational cubic equation which shall have for roots $1, 3 - \sqrt{-2}$.

Solution.

Since $3 - \sqrt{-2}$ is a root of the equation, $3 + \sqrt{-2}$ is also a root. So we

have to form an equation whose roots are $1, 3 - \sqrt{-2}, 3 + \sqrt{-2}$.

Hence the required equation is $(x - 1)(x - 3 - \sqrt{-2})(x - 3 + \sqrt{-2}) = 0$

$$(x - 1)\{(x - 3)^2 + 2\} = 0$$

$$(x - 1)(x^2 - 6x + 11) = 0$$

$$x^3 - 7x^2 + 17x - 11 = 0.$$

2. Solve the equation $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$ of which one root is $-1 + \sqrt{-1}$.

Solution.

Imaginary roots occur in pairs. Hence $-1 - \sqrt{-1}$ is also a root of the equation.

Therefore the expression on the left side of equation has the factors

$(x + 1 - \sqrt{-1})(x + 1 + \sqrt{-1})$.

The expression on the left side is exactly divisible by $(x + 1)^2 + 1$, i.e.,$x^2 + 2x + 2$.

Dividing $x^4 + 4x^3 + 5x^2 + 2x - 2$ by $x^2 + 2x + 2$, we get the quotient $x^2 + 2x - 1$.

Therefore $x^4 + 4x^3 + 5x^2 + 2x - 2 = (x^2 + 2x + 2)(x^2 + 2x - 1)$.

Hence the other roots are obtained from $x^2 + 2x - 1 = 0$.

Thus the other roots are $-1 \pm \sqrt{2}$.

3. Show that $\frac{a^2}{x-\alpha}+\frac{b^2}{x-\beta}+\frac{c^2}{x-\gamma} - x + \delta = 0$ has only real roots if a, b, c, α, β, γ, δ are real.

Solution.

If possible let p + iq be a root. Then p – iq is also root.

Substituting these values for x, we have

$$\frac{a^2}{p+iq-\alpha}+\frac{b^2}{p+iq-\beta}+\frac{c^2}{p+iq-\gamma}-p-iq+\delta=0 \quad ......(1)$$

$$\frac{a^2}{p-iq-\alpha}+\frac{b^2}{p-iq-\beta}+\frac{c^2}{p-iq-\gamma}-p+iq+\delta=0 \quad ......(2)$$

Substituting (2) from (1), we get

$$-\frac{2a^2iq}{(p-\alpha)^2+q^2}-\frac{2b^2iq}{(p-\beta)^2+q^2}-\frac{2c^2iq}{(p-\gamma)^2+q^2}-2iq=0$$

$$-2iq\left\{\frac{a^2}{(p-\alpha)^2+q^2}+\frac{b^2}{(p-\beta)^2+q^2}+\frac{c^2}{(p-\gamma)^2+q^2}+1\right\}=0$$

This is only possible when q = 0 since the other factor cannot be zero. In that case the roots are real.

In an equation with rational coefficients irrational roots occur in pairs.

Let f(x) = 0 denotes the equation and suppose that a + $\sqrt{b}$ is a root of the equation where a and b are rational and $\sqrt{b}$ is irrational. We now show that a – $\sqrt{b}$ is also a root of the equation

$$(x-a-\sqrt{b})(x-a+\sqrt{b}) = (x-a)^2 - b \quad ....(1)$$

If f(x) is divided by $(x-a)^2 - b$, let the quotient be Q(x) and the remainder be Rx + R'.

Here Q(x) is a polynomial of degree (n – 2).

$\therefore$ f(x) = {$(x-a)^2 - b$} Q(x) + Rx + R' ....(2)

Substituting a + $\sqrt{b}$ for x in (2), we get

$$f(a+\sqrt{b}) = \{(a+\sqrt{b}-a)^2 - b\}\, Q(a+\sqrt{b}) + R(a+\sqrt{b}) + R'$$

$$= R(a+\sqrt{b}) + R'$$

but $f(a+\sqrt{b}) = 0$, since $a+\sqrt{b}$ is a root of $f(x) = 0$.

$\therefore\ Ra + R' + R\sqrt{b} = 0.$

Equating the rational and irrational parts, we have

$Ra + R' = 0$ and $R = 0$.

$\therefore R' = 0.$

Hence $f(x) = \{(x-a)^2 - b\}Q(x).$

$= (x - a - \sqrt{b})(x - a + \sqrt{b})Q(x).$

$\therefore\ a - \sqrt{b}$ is a root of $f(x) = 0$.

Solved Problems

Example 1. Frame an equation with rational coefficients, one of whose root is $\sqrt{5}+\sqrt{2}$

Solution.

Then the other roots are $\sqrt{5}-\sqrt{2}, -\sqrt{5}+\sqrt{2}, -\sqrt{5}-\sqrt{2}$

Hence the required equation is $(x-\sqrt{5}-\sqrt{2})(x-\sqrt{5}+\sqrt{2})(x+\sqrt{5}+\sqrt{2})(x+\sqrt{5}-\sqrt{2}) = 0$

i.e. $\{(x-\sqrt{5})^2 - 2\}\{(x+\sqrt{5})^2 - 2\} = 0$

i.e. $(x^2 - 2x\sqrt{5} + 3)(x^2 + 2x\sqrt{5} + 3) = 0$

i.e. $(x^2+3)^2 - 4x^2.5 = 0$

i.e. $x^4 - 14x^2 + 9 = 0.$

Example 2. Solve the equation $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$ given that one of the roots is $1 - \sqrt{5}$.

Solution.

Since the irrational roots occur in pairs, $1 + \sqrt{5}$ is also a root. The factors corresponding to these roots are

$(x - 1 + \sqrt{5})(x - 1 - \sqrt{5})$, i.e. $(x - 1)^2 - 5$

i.e. $x^2 - 2x - 4$.

Dividing $x^4 - 5x^3 + 4x^2 + 8x - 8$ by $x^2 - 2x - 4$, we get the quotient $x^2 - 3x + 2$.

Therefore $x^4 - 5x^3 + 4x^2 + 8x - 8 = (x^2 - 2x - 4)(x^2 - 3x + 2)$

$$= (x^2 - 2x - 4)(x - 1)(x - 2)$$

The roots of the equation are $1 \pm \sqrt{5}$, 1, 2.

Example 3. Form the equation with rational coefficients whose roots are

(i) $1 + 5\sqrt{-1}, 5 - \sqrt{-1}$

(ii) $-\sqrt{3} + \sqrt{-2}$.

Solution :

(i) $1 + 5\sqrt{-1}, 5 - \sqrt{-1}$

Then the other roots are $1 + 5\sqrt{-1}, 5 - \sqrt{-1}, 1 - 5\sqrt{-1}, 5 + \sqrt{-1}$

Hence the equation is

$$(x - 1 + 5\sqrt{-1})(x - 1 - 5\sqrt{-1})(x - 5 - \sqrt{-1})(x - 5 + \sqrt{-1}) = 0$$

$$\{(x - 1)^2 - (5\sqrt{-1})^2\}\left\{(x - 5)^2 - \left(\sqrt{-1}\right)^2\right\} = 0$$

$$(x^2 - 2x + 26)(x^2 - 10x + 26) = 0$$

$$x^4 - 12x^3 + 72x^2 - 312x + 676 = 0.$$

(ii) $-\sqrt{3} + \sqrt{-2}$

Then the other roots are $-\sqrt{3} + \sqrt{-2}, -\sqrt{3} - \sqrt{-2}, \sqrt{3} + \sqrt{-2}, \sqrt{3} - \sqrt{-2}$

$$\left\{\left(x + \sqrt{3}\right)^2 - \left(\sqrt{-2}\right)^2\right\}\left\{\left(x - \sqrt{3}\right)^2 - \left(\sqrt{-2}\right)^2\right\} = 0$$

$$(x^2 + 2\sqrt{3}x + 5)(x^2 - 2\sqrt{3}x + 5) = 0$$

$$x^4 - 2x^2 + 25 = 0.$$

Example 4. Solve : $x^4 - 4x^3 + 8x + 35 = 0$ given that $2 + i\sqrt{3}$ is a root of it.

Solution.

Since the irrational roots occur in pair, $2 - i\sqrt{3}$ is also a root.

The factors corresponding to these roots are $(x - 2)^2 - (i\sqrt{3})^2$

$$x^2 - 4x + 7.$$

Dividing $x^4 - 4x^3 + 8x + 35$ by $x^2 - 4x + 7$, we get the equation $x^2 + 4x + 5$

$$x^4 - 4x^3 + 8x + 35 = (x^2 - 4x + 7)(x^2 + 4x + 5)$$

The roots of the equation are $2 \pm i\sqrt{3}, -2 \pm i$

Example 5.Solve the equation $2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81 = 0$ given that one root is $\sqrt{2} - \sqrt{-1}$.

Solution.

Then the other roots are $\sqrt{2} - \sqrt{-1}, \sqrt{2} + \sqrt{-1}, -\sqrt{2} - \sqrt{-1}, -\sqrt{2} + \sqrt{-1}$

$$\left\{(x - \sqrt{2})^2 - (\sqrt{-1})^2\right\}\left\{(x + \sqrt{2})^2 - (\sqrt{-1})^2\right\} = 0$$

$$(x^2 - 2\sqrt{2}x + 3)(x^2 + 2\sqrt{2}x + 3) = 0$$

$$x^4 - 2x^2 + 9 = 0$$

Dividing $2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81$ by $x^4 - 2x^2 + 9$ we get the equation $2x^2 - 3x + 9$

$$2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81 = (x^4 - 2x^2 + 9)(2x^2 - 3x + 9)$$

The roots of the equation are $\sqrt{2} \pm \sqrt{-1}, , -\sqrt{2} \pm \sqrt{-1}, 3\left(\frac{1 \pm i\sqrt{7}}{4}\right)$